Physical Applications of Products of Geometric Distributions.
Phys 451: Masters project.

Calum Ross.
Supervisor: Jonathan Gratus.

May 1, 2014

Abstract

The mathematical techniques used in electromagnetism have applications to many other areas of physics. This makes it a useful area to test new mathematical approaches. In this case, that approach is an extension of the method of Green’s functions to the setting of coordinate free differential geometry. This would provide a novel way to tackle some problems, such as the monopole, as well as offering an approach to others that can’t currently be solved in a satisfactory manner, the dipole. To achieve this distributions are introduced in a coordinate free manner, followed by an introduction to some of their properties. Their interaction with certain key concepts from differential geometry, such as the pushforward and pullback maps as well as the wedge product is also dealt with. This leads to the main aim, a coordinate free treatment of Green’s functions, including the calculation of the monopole potential, the Liénard-Wiechert potential, which is used as the benchmark result to show that the techniques work. Following on from this, the open problem of a dipole source is explored and a possible solution is conjectured. Finally, future work including fixing the calculation for the dipole and extensions to higher orders is discussed.
## Contents

1 Introduction 3

2 Distributions 4
   2.1 Interaction of Distributions with $d$, $i_v$ and $*$ 6

3 Pushforwards of Distributions 6
   3.1 Pushforward and $d$ 8
   3.2 Pushforward and $i_v$ 10

4 The Wedge Product and Distributions 11

5 Pullbacks of Distributions, Definitions 14
   5.1 Setup 14
   5.2 Proof of Equivalence 15
   5.3 Relation of Distributional Pullback to $d$, $i_v$ and $*$ 17
      5.3.1 Uniqueness of Vectors 19

6 The $\Delta$ Conjecture 20
   6.1 Setup 20
   6.2 Proof of the $\Delta$ conjecture for $\Psi = a_v \alpha$ 21
   6.3 Proof of the $\Delta$ conjecture for $\Psi = i_v a_c \alpha$ 22

7 Green’s functions 23
   7.1 Green’s Functions in Electrodynamics 24

8 The Dipole 30

9 Conclusion 32

A Basic Properties of Distributions 33
1 Introduction

The theory of electromagnetism is something of a showcase for interesting mathematical techniques used throughout physics. This makes it the ideal testbed for exploring the implications of some new techniques. In this case we will be looking at extensions to the standard linear distribution theory; of particular interest is extending some of the techniques used for solving the differential equations describing certain physical problems. The method followed in this thesis is to develop the necessary techniques in a more abstract setting before bringing them together to be applied to a physical problem.

For problems in electromagnetism the first step is to try and solve Maxwell’s equations in the desired setting. As Maxwell’s equations are second order linear inhomogeneous partial differential equations, there are several ways that solving them can be attempted. However, the most common approach would be to use the method of Green’s functions. Green’s functions work by changing the equation that you are looking at for an analogous, simpler equation for the Green’s function. This analogous equation is normally easier to solve; usually this is just a case of applying another mathematical technique, the Fourier transform. Once it has been found, the solution, referred to as the Green’s function, can be used to solve the original equation. This technique sees widespread use all over physics for problems as diverse as solving wave equations, which is what is of interest here, to calculating Feynman propagators in quantum field theory.

There is one slight problem with the current Green’s function techniques, which isn’t a problem for most applications but is more of a desirable extension. This is that they can’t, currently, be interpreted in the coordinate free language of differential geometry. This causes some stumbling blocks as problems can be more easily expressed using this language, particularly those from electromagnetism and general relativity. The main aim of this project was to get a working extension of this method to the coordinate free setting. To achieve this a lot of ground work needed to be done to extend techniques from standard differential geometry to the setting of distributions, Sections 2 to 6. Distributions, introduced in Section 2, are the key objects for extending the Green’s function methods in the desired manner. However, the standard approach to distributions has a major stumbling block in the form of the Schwartz impossibility result [1], which says that in general there is no product on the space of distributions. There have been several attempts to get around this, for a discussion of these methods see the review [2]. In this thesis there will be no attempt to follow any of these approaches, though a brief review is given in Section 4. What will be done is to restrict interest to a subspace of distributions where products can be taken, in fact this was the focus of a Nuffield undergraduate research bursary over the summer of 2012, and a discussion of this is the main aim of Section 4. Some of the standard tools from differential geometry, pushforwards and pullbacks of maps, can and are extended to the case of distributions; this is the purpose of Sections 3 and 5. Once there are some tools in the box, a rather nice conjecture could be formulated and proved in certain circumstances, Section 6. This conjecture, as innocuous as it may seem, is a key step in getting the Green’s function techniques to work in the desired manner. The specifics of
this conjecture, referred to as the $\Delta$ conjecture, relates to the product of a distribution with the $\Delta$ distribution. In words it states that if a distribution is pulled back on to a product manifold, then wedged with the $\Delta$ distribution and finally pushed forward onto the other part of the product manifold; the result is the original distribution. The reason this is closely related to the Green’s function results is that the differential equation for the Green’s function is given as a differential operator acting on the Green’s function being equal to the $\Delta$ distribution.

The construction of the Green’s function techniques is what the first part of Section 7 is devoted to, with the latter part focusing on electrodynamics and the solution of Maxwell’s equations for a monopole. A monopole is a point object with a single charge, it is the simplest object to solve the equations of electrodynamics for; an electron is sometimes pictured as being one. The calculation of this solution, known as the Liénard-Wiechert potential, is the test of the new methods suggested. Following on from replicating and discussing the standard result an open problem is discussed, Section 8. This is the dipole, which in terms of electric charge would be an object possessing both positive and negative charge. For an object of finite extent this can be thought of as two charges joined by a rod of some kind. However, for the case of a point object nobody really knows how to think about the dipole, if it even makes sense for electric charge. For magnetic charge this could be visualised as being an electron again. Though a conclusive calculation for the dipole is still elusive, it is hoped that the gaps in the current approach can be filled in soon and that after this a satisfactory solution can be obtained.

2 Distributions

A key difference between the treatment of distributions given here, as apposed to elsewhere, is that they will be formulated in the language of differential geometry right from the start. This treatment was first suggested by De Rham,[3], and is a very useful approach to problems involving distributions as it gives a new way of considering the problems. This also leads to new applications of distributions to physical problems. As such a basic understanding of differential geometry around the level presented in Taubes [4] is assumed, a summary can be found in this review [2]. A distribution can be thought of as a map from a subspace of the functions to the real numbers; this subspace is called the test functions. The first thing to do is to define the test functions as test forms over a differentiable manifold.

**Definition 2.0.1.** $\Gamma_0\Lambda^p M$ denotes the space of test forms over $M$ with compact support. The test forms have the following properties;

1. They are only non-zero inside a bounded, finite interval. This property is called compact support.
2. They are infinitely differentiable (smooth).

This leads to the definition of a distribution that will be assumed throughout.

**Definition 2.0.2.** A Distribution is a linear functional, $\Psi$, which acts on test forms in the following way, $\Psi : \Gamma_0\Lambda^p M \rightarrow \Gamma_0\Lambda^p M$. Or in terms of action on a specific test form $\phi \in \Gamma_0\Lambda^p M$ such that $\Psi : \phi \mapsto \langle \phi, \Psi \rangle_M$. 

4
The linear part means that $\forall \lambda \in \mathbb{R}$, $\langle \phi_1 + \lambda \phi_2, \Psi \rangle_M = \langle \phi_1, \Psi \rangle_M + \lambda \langle \phi_2, \Psi \rangle_M$. Where the angled bracket notation is used to denote the action of a distribution on a test function, it was picked to fit in nicely with the integral interpretation of the action of a regular distribution that is used below. Another benefit of this notation is that the subscript on the brackets makes it easy to keep track of which manifold is being worked on; this will prove invaluable when dealing with maps between manifolds. The big step in interpreting distributions using differential geometry is to treat them as $p$-form distributions.

**Definition 2.0.3.** The space of all $p$-form distributions over a differentiable manifold $M$ is denoted $\Gamma_D \Lambda^p M$.

As well as the very general distributions defined above, there is a specific subset called the regular distributions, these can be represented in terms of an integral.

**Definition 2.0.4.** A distribution, $\alpha^D \in \Gamma_D \Lambda^p M$, is called a regular distribution if $\exists \alpha \in \Gamma \Lambda^p M$ such that,

$$\langle \phi, \alpha^D \rangle_M = \int_M \phi \wedge \alpha.$$  \hfill (2.1)

The most useful class of distributions is closely related to the regular distributions, they are a subset of the regular distributions called submanifold distributions. These are distributions which are built from regular distributions through the action of $d$, $i_v$ and $\star$.

**Definition 2.0.5.** The space of $p$-form submanifold distributions on $M$ is denoted $\Gamma_{SD} \Lambda^p M$.

It is these submanifold distributions that will be used throughout the thesis. There is also the concept of a degree for distributions, analogous to the concept of a degree for a form. For forms the degree can, naively, be interpreted as the number of 1-forms wedged together. This leads to an obvious definition.

**Definition 2.0.6.** For $\alpha \in \Gamma \Lambda^p M$ the degree of $\alpha$ is given by $|\alpha|$ and $|\alpha| = p$.

This definition can be extended to distributions.

**Definition 2.0.7.** For $\Psi \in \Gamma_D \Lambda^p M$ the degree of $\Psi$ is given by $|\Psi|$ and $|\Psi| = p$.

The concept of the degree of a distribution is useful when considering its action on a test form, which also has a degree through Definition 2.0.6. The natural result is that $\langle \phi, \Psi \rangle_M = 0$ when $|\phi| + |\Psi| \neq \dim M$. This arises due to $\langle \phi, \Psi \rangle_M$ being equivalent to an integral over $M$. Another useful concept is that of the support of a distribution.

**Definition 2.0.8.** For $\Psi \in \Gamma_D \Lambda^p M$ the support of $\Psi$ is the subset of $M$ for which $\langle \phi, \Psi \rangle_M \neq 0 \forall \phi \in \Gamma_0 \Lambda^{m-p} M$, where $m = \dim M$.

The fact that a distribution has a region of support, which need not be the entire manifold, means that in some cases calculations can be simplified by working on this region of support. Now that the requisite definitions are out of the way, it is time to consider some of the properties of the distributions and how they interact with key objects from differential geometry.
2.1 Interaction of Distributions with $d$, $i_v$ and $\star$

**Theorem 1.** For a distribution $\Psi \in \Gamma_D \Lambda M$, a vector $v \in \Gamma T M$ and a test function $\phi \in \Gamma_0 \Lambda M$. There are a set of relationships for how $d$, $i_v$ and $\star$ can hop across from the distribution to the test function.

- **Hopping of the exterior derivative,**
  \[ \langle \phi, d\Psi \rangle_M = (-1)^{1-|\phi|} \langle d\phi, \Psi \rangle_M. \]  
  (2.2)

- **Hopping of the internal contraction,**
  \[ \langle \phi, i_v \Psi \rangle_M = (-1)^{1-|\phi|} \langle i_v \phi, \Psi \rangle_M. \]  
  (2.3)

- **Hopping of the Hodge star,**
  \[ \langle \phi, \star \Psi \rangle_M = (-1)^{|\phi|} |\star \phi| \langle \star \phi, \Psi \rangle_M. \]  
  (2.4)

Note that $1 - |\phi| = |\Psi| - n$ where $n = \dim M$. The proof of the above statements is included in Appendix A.

3 Pushforwards of Distributions

The notion of a pushforward can be extended to work with distributions, and as with most properties of distributions the explicit form is defined through the action on a test form.

**Definition 3.0.1.** Take two manifolds $M$ and $N$ and a proper map, $f : N \to M$, between them; then the distributional pushforward is given by $f_* : \Gamma_D \Lambda N \to \Gamma_D \Lambda M$.

A proper map is a map which takes compact sets to compact sets. For a specific $\Psi \in \Gamma_D \Lambda N$, $f_* \Psi \in \Gamma_D \Lambda M$ is the pushforward of $\Psi$.

**Definition 3.0.2.** For $\Psi \in \Gamma_D \Lambda N$ and $\phi \in \Gamma_0 \Lambda M$, the pushforward of a distribution, by a proper map $f : N \to M$ is linked to the standard pullback of a form through,

\[ \langle \phi, f_* \Psi \rangle_M = \langle f^* \phi, \Psi \rangle_N. \]  
(3.1)

The pushforward can be visualised with the help of Figure 1. Figure 2 gives another visualisation of a pushforward, though the distribution pictured is an example of one that is best avoided. This is a good point to resume the discussion of the concept of the degree of a distribution. Now that the pushforward has been introduced, there are two types of degree that a distribution of the form $\Psi = a \cdot \alpha$ can have. These are the standard degree, introduced above, $|\Psi|$, and the internal degree $\text{ideg}(\Psi) = |\alpha|$. The internal degree will be very useful later on when products of distributions are being taken, in fact the internal degree leads to the algebra developed later being referred to as an internal graded algebra. Another useful fact about the degree of a test form is that the pullback preserves degrees, this means that $|f^* \phi| = |\phi|$. 

6
If the map, $f$, is not proper then a partition of unity, $\psi_i$, needs to be used. A partition of unity a family of functions $\psi_i$ where $i$ is in some, possibly infinite, indexing set $I$ such that $\text{supp}\psi_i$ is compact, $\sum_{i \in I} \psi_i = 1$ and $\psi_i|_x \neq 0$ for only a finite number of $\psi_i$’s. An example of a partitioned manifold is given in Figure 3. Using the partition of unity Definition 3.0.2 becomes;

**Definition 3.0.3.** For $\Psi \in \Gamma_D \Lambda N$ and $\phi \in \Gamma_0 \Lambda M$ the pushforward of a distribution, by $f : N \to M$ is linked to the standard pullback of a test form through,

$$\langle \phi, f_\ast \Psi \rangle_M = \sum_{i \in I} \langle \psi_i f^\ast \phi, \Psi \rangle_N.$$  \hfill (3.2)

This is not always defined unfortunately, it depends on what $\Psi$ is as well as what $N$ and $M$ are. The problems occur when $f$ is a projection, and the pushforward can be identified with an integral over the coordinates being projected out. For example take $\Psi = 1^D \in \Gamma_D \Lambda^1 \mathbb{R}^2$, $N = \mathbb{R}^2$ and $M = \mathbb{R}$, so $f : \mathbb{R}^2 \to \mathbb{R}$. Here

$$f_\ast(1^D) = \int_\mathbb{R} 1 \, dx,$$  \hfill (3.3)
Figure 2: Another example of the pushforward of a distribution. In this case the map $f$ is a projection. This is also an example of a situation it is best to avoid. Though the pushforward is defined, due to the nature of $\Psi$ it is unknown if $f_*\Psi$ can be wedged with other distributions. This is because the right hand limit point of $f_*\Psi$ can cause continuity problems.

which is infinite and hence the pushforward can’t be defined in the manner above. It should be noted that this definition is expected to be independent of the partition of unity, this is due to the analogous result that integrals are independent of the chosen partition of unity. Now that the pushforward of a distribution has been defined, we can start to look at it’s properties such as how it interacts with $d$’s, $i_v$’s and eventually wedge products.

The following results were found during the course of a Nuffield research bursary over the summer of 2012; they are included for completeness and because they will be useful in later sections.

### 3.1 Pushforward and $d$

Now that we have Definitions 3.0.2 and 3.0.3 we can start to consider expressions of the form

$$\langle \phi, f_*d\Psi \rangle_M.$$  \hspace{1cm} (3.4)

Where $f : M \to N$ is a proper map, $m = \dim M$, $\Psi \in \Gamma_D \Lambda N$, $f_*d\Psi \in \Gamma_D \Lambda^{p+1}M$ and $\phi \in \Gamma_0 \Lambda^{m-p-1}M$. To see how the $d$ interacts with the pushforward it is easiest to start
Figure 3: An example of a partitioned manifold, each numbered region would have a different partition of unity $\psi_i$ for $i \in \{1, 2, 3, 4\}$.

labeling the $d$ with which manifold it is on, so $d_N$ is the exterior derivative on $N$ and $d_M$ is the same on $M$. Now we are ready to state a theorem about the pushforward.

**Theorem 2.** For a map, $f : N \to M$, the exterior derivative commutes with the pushforward.

$$\langle \phi, f_* d_N \Psi \rangle_M = \langle \phi, d_M f_* \Psi \rangle_M$$

(3.5)

Alternatively this can be written as,

$$f_* \circ d_N = d_M \circ f_*.$$  

(3.6)

**Proof.** First we shall deal with the case when $f$ is a proper map. The proof starts with the left hand side and then applies the standard manipulations of the pushforward and
the pullback.

\[
\langle \phi, f_{s}d_N \Psi \rangle_M = (f^* \phi, d_N \Psi)_N \\
= (-1)^{1-|\phi|} (d_N f^*(\phi), \Psi)_N \\
= (-1)^{1-|\phi|} (f^*(d_M(\phi)), \Psi)_N \\
= (-1)^{1-|\phi|} (d_M(\phi), f_{s}\Psi)_M \\
= (-1)^{1-|\phi|}(-1)^{1-|\phi|}(\phi, d_M f_{s}\Psi)_M \\
= \langle \phi, d_M f_{s}\Psi \rangle_M
\]

Now for the non proper map everything is fairly similar.

\[
\langle \phi, f_{s}d_N \Psi \rangle_M = \sum_{i \in I} \langle \psi_i f^* \phi, d_N \Psi \rangle_N \\
= (-1)^{1-|\phi|} \sum_{i \in I} \langle d_N(\psi_i f^*(\phi)), \Psi \rangle_N \\
= (-1)^{1-|\phi|} \sum_{i \in I} \langle [d_N\psi_i \wedge f^*(\phi) + \psi_i f^*(d_M(\phi))], \Psi \rangle_N
\]

To resolve this we need to use that \(\sum_{i \in I} \langle \phi_i, \Psi \rangle_N = \langle \sum_{i \in I} \phi_i, \Psi \rangle_N\), this is true whenever the right hand side is defined, or in other words whenever \(\sum_{i \in I} \phi_i\) is a test form. It holds in the above due to the properties of the partition of unity. This is not quite the end of the story, there is still the \(\sum_{i \in I} (d_N\psi_i \wedge f^*(\phi))\), \(\Psi\) \(N\), term to deal with. Here we need the result that for a test form the derivative is bounded, this allows the sum to be moved through the derivative giving, \(\sum_{i \in I} (d_N\psi_i \wedge f^*(\phi)) = (d_N \sum_{i \in I} \psi_i \wedge f^*(\phi))\). The final step is to use \(\sum_{i \in I} \psi_i = 1\), which implies \(d_N \sum_{i \in I} \psi_i = d_N 1 = 0\). This means that \(\sum_{i \in I} (d_N\psi_i \wedge f^*(\phi)) = 0\). Substituting this into the result above gives,

\[
\langle \phi, f_{s}d_N \Psi \rangle_M = (-1)^{1-|\phi|} \sum_{i \in I} \langle 0 + \psi_i d_N f^*(\phi), \Psi \rangle_N \\
= (-1)^{1-|\phi|} \langle d_M(\phi), f_{s}\Psi \rangle_M \\
= (-1)^{1-|\phi|}(-1)^{1-|\phi|} \langle \phi, d_M f_{s}\Psi \rangle_M \\
= \langle \phi, d_M f_{s}\Psi \rangle_M
\]

Hence the desired result is true even for non proper maps. \(\square\)

So the commutativity of the exterior derivative with the pullback leads directly to its commutativity with the pushforward as well.

### 3.2 Pushforward and \(i_{v}\)

There is an analogous relation for how an internal contraction interacts with the pushforward. However, this is slightly more complicated than the expression for \(d\). In this case, we will be dealing with expressions of the form

\[
\langle \phi, f_{s}i_{v} \Psi \rangle_M,
\]

(3.7)
where everything has the same meaning as before, \( m = \dim M, \Psi \in \Gamma D \Lambda N, f_s i_v \Psi \in \Gamma_D \Lambda^{p-1} M \) and \( \phi \in \Gamma_0 \Lambda^{m-p+1} N, \) and \( v \in \Gamma TN. \)

**Theorem 3.** For a proper map, \( f : N \to M, \) and vectors, \( w \in \Gamma TM \) and \( v \in \Gamma TN, \) such that \( w|f(p) = f_s(v)|_p \), or in other words the pushforward of \( v \) is defined. In this case the pushforward will commute with the internal contraction.

\[
\langle \phi, f_s i_w \Psi \rangle_N = \langle \phi, i_v f_s \Psi \rangle_N.
\] (3.8)

Alternatively,

\[
f_c \circ i_w = i_v \circ f_c. \] (3.9)

**Proof.** To prove this we just start with the left hand side and proceed using the known manipulations.

\[
\langle \phi, f_s i_w \Psi \rangle_N = \langle f^* \phi, i_w \Psi \rangle_M \\
= (-1)^{1-|\phi|} \langle i_w f^* \phi, \Psi \rangle_M \\
= (-1)^{1-|\phi|} \langle i_w f^* \phi, \Psi \rangle_M \\
= (-1)^{1-|\phi|} \langle f^*(i_v \phi), \Psi \rangle_M \\
= (-1)^{1-|\phi|} \langle i_v \phi, f_c \Psi \rangle_N \\
= (-1)^{1-|\phi|} \langle f_c \circ i_w = i_v \circ f_c. \]

This will also hold for some non-proper maps, in a similar manner to the result for \( d \) above. However, this requires that \( f_s v \) is defined which it need not be. This means that we can now deal with a wide variety of distributions which involve combinations of \( d \)'s and \( i_v \)'s. The next step is to consider wedge products of distributions.

### 4 The Wedge Product and Distributions

This section is included as a review of some of the results found during a Nuffield summer research project, it contains some ideas that are useful when it comes to considering the dipole, Section 8.

In the standard linear, Schwartz, theory of distributions there is no product on the space of distributions. This was shown by the famous Schwartz impossibility result [1]. There have been several attempts to get around this by either tweaking the requirements on the product [1], or changing the space of distributions. The most popular approach follows the latter tactic and is the Colombeau theory of generalized functions [5]. In the following there is no attempt to directly replicate either of these procedures; distributions will be treated in the standard sense with the only difference being they will be expressed in the language of differential geometry. This step of looking at distributions in a new light is surprisingly powerful. It allows a formal criteria for when
products of certain distributions can be taken. This criteria can be formulated with the help of the following diagram.

\[
\begin{array}{ccc}
C & \xrightarrow{\hat{a}} & A \\
\downarrow{c} & & \downarrow{a} \\
B & \xleftarrow{\hat{b}} & M
\end{array}
\]

The hooked arrows signify regular immersions, injective maps between manifolds, this is because \( \dim A \leq \dim M \) and \( A \subseteq M \). For the map \( a : A \to M \), being a regular immersion can be understood in the following manner. \( \forall x \in M, \exists \) a coordinate system \((x^1, \ldots, x^m) \) on \( M \) and \((y^1, \ldots, y^n) \) on \( A \), where \( m \geq a \) such that \( a(y^1, \ldots, y^n) = (y^1, \ldots, y^a, 0, \ldots, 0) \). In the above diagram \( C = \{(w, z) \in A \times B | a(w) = b(z)\} = A \cap B \) and \( A \cup B \subset M \). This condition on \( C \) is basically saying that \( A \) and \( B \) intersect transversely and \( C \) is the manifold defined by their intersection. Now the conditions to “intersect transversely” can be summed up by considering coordinates again. These coordinates are \((w^1, \ldots, w^r, y^{r+1}, \ldots, y^a) \) on \( A \), \((w^1, \ldots, w^r, z^{r+1}, \ldots, z^a) \) on \( B \). As \( C = A \cap B \) the coordinates on \( C \) are given by \((w^1, \ldots, w^r) \), and finally those on \( M \) are \((x^1, \ldots, x^m) \). All these coordinates are related by, \( a(w^1, \ldots, w^r, y^{r+1}, \ldots, y^a) = (w^1, \ldots, w^r, y^{r+1}, \ldots, y^a, 0, \ldots, 0) \) and \( b(w^1, \ldots, w^r, z^{r+1}, \ldots, z^a) = (w^1, \ldots, w^r, 0, \ldots, 0, z^{r+1}, \ldots, z^b) \). These coordinate relations can be used as a definition of transverse intersection. An alternative definition is in terms of the tangent bundles \( TA \) and \( TB \) and involves saying that they are locally perpendicular. This means that a vector in a neighborhood of the intersection can be decomposed into \( v_A + v_B \) where \( v_A \in TA \) and \( v_B \in TB \) [4]. So our criteria for taking products is when \( A \) and \( B \) are transverse. Now to the actual mechanics of taking products, the trick is that the distributions need to be on different manifolds, that is one is on \( A \) and the other on \( B \), then they can both be pushed forward on to \( M \) where they are wedged together. This has only been done with submanifold distributions so far, though it may apply more generally. Lets demonstrate this for the simplest case, \( \alpha \in \Gamma_{SD} A, \beta \in \Gamma_{SD} B \).

**Definition 4.0.1.** The wedge product of two regular distributions, from \( A \) and \( B \) respectively, on \( M \) can be interpreted through first sending them to \( C \) before wedging them, promoting them to distributions then finally pushing them forward to \( M \). This is summarised as,

\[
\alpha_\varsigma(\alpha) \wedge b_\varsigma(\beta) = c_\varsigma(\hat{a}^*(\alpha) \wedge \hat{b}^*(\beta)).
\] (4.1)

This is essentially just a rewriting using the above diagram. However it makes a big difference when thinking about the action on a test form as we now have the product in terms of one distribution on \( C \). This acts on a test form as follows.

\[
\langle \phi, \alpha_\varsigma(\alpha) \wedge b_\varsigma(\beta) \rangle_M = \langle \phi, c_\varsigma(\hat{a}^*(\alpha) \wedge \hat{b}^*(\beta)) \rangle_M = \langle c^*(\phi), \hat{a}^*(\alpha) \wedge \hat{b}^*(\beta) \rangle_C = \int_C c^*(\phi) \wedge \hat{a}^*(\alpha) \wedge \hat{b}^*(\beta)
\]
So for this relatively simple case, it was easy to get an integral out representing the action of the product of the two distributions on a test function. The next logical step is to ask about what happens when looking at more general submanifold distributions on $M$. Examples of these would be,

1. $d(a_c(\alpha) \wedge b_\zeta(\beta))$,
2. $i_v(a_c(\alpha) \wedge b_\zeta(\beta))$, where $v \in TM$,
3. $a_c(\alpha) \wedge i_v b_\zeta(\beta)$ and
4. $d(i_v a_c(\alpha) \wedge b_\zeta(\beta))$.

The first two can be dealt with in the same way as above because they are acting on the whole distribution. The interesting things starts to happen when number 3 is considered. Here the contraction is only acting on one distribution, and this is where the transversality of $A$ and $B$ comes in, and as mentioned above this means that locally a vector can be decomposed into one part on $TA$ and another part on $TB$. In other words, $v = a_\ast(v_a) + b_\ast(v_b)$, this coupled with the fact that $i_{b_\ast(v_b)}$ can pass through $b_\zeta$ to make sense of the third example.

$$a_c(\alpha) \wedge i_v b_\zeta(\beta) = a_c(\alpha) \wedge i_{a_\ast(v_a)+b_\ast(v_b)} b_\zeta(\beta)$$

$$= a_c(\alpha) \wedge i_{a_\ast(v_a)} b_\zeta(\beta) + a_c(\alpha) \wedge i_{b_\ast(v_b)} b_\zeta(\beta)$$

$$= a_c(\alpha) \wedge i_{a_\ast(v_a)} b_\zeta(\beta) + a_c(\alpha) \wedge b_\zeta(i_v \beta)$$

The first part can also be evaluated in a similar way using the fact that,

$$i_v(a_c(\alpha) \wedge b_\zeta(\beta)) = i_v a_c(\alpha) \wedge b_\zeta(\beta) + (-1)^{\text{ideg}(a_c(\alpha))} a_c(\alpha) \wedge i_v b_\zeta(\beta).$$

So there are three terms that come out of 3 but fortunately we know how to deal with all of them. This means that the first interesting case can be dealt with given an internal graded algebra similar to that of forms. The next question to ask is “can Example 4 be dealt with in the same way?” There isn’t really a simple answer to this, in fact this is one of the open problems that needs to be solved to complete the dipole calculation, see Section 8. The confusion comes because the same trick as was used for Example 3 will only go so far. Using it can get us to

$$d(i_v a_c(\alpha) \wedge b_\zeta(\beta)) = d(i_{a_\ast(v_a)} a_c(\alpha) \wedge b_\zeta(\beta) + i_{b_\ast(v_b)} a_c(\alpha) \wedge b_\zeta(\beta)),$$

$$= d(a_c(i_v a_\ast(\alpha) \wedge b_\zeta(\beta) + i_{b_\ast(v_b)} a_c(\alpha) \wedge b_\zeta(\beta)) - (-1)^{\text{ideg}(a_c(\alpha))} a_c(\alpha) \wedge b_\zeta(i_v \beta)).$$

This would seem to be fine as the $d$ will hop over on to the test function and all the contractions are in a form that can be dealt with. However, this is only one possible way to expand the expression. If instead the $d$ is brought in first, a very interesting term appears.

$$d(i_v a_c(\alpha) \wedge b_\zeta(\beta)) = d(a_c(i_v a_\ast(\alpha) \wedge b_\zeta(\beta)) + d i_{b_\ast(v_b)} a_c(\alpha) \wedge b_\zeta(\beta) + (-1)^{\text{ideg}(a_c(\alpha)) - 1} i_{b_\ast(v_b)} a_c(\alpha) \wedge d b_\zeta(\beta).$$

The interesting term is $d i_{b_\ast(v_b)} a_c(\alpha) \wedge b_\zeta(\beta)$, here the contraction is trapped behind the $d$. This means that the action on a test form of this term is undefined and cannot be dealt with directly. A possible way around this would be to define $d i_{b_\ast(v_b)} a_c(\alpha) \wedge b_\zeta(\beta)$ in terms of quantities that can be dealt with. However, this needs slightly more work.
to check that the expression that comes out has the correct properties. It wasn’t felt that there was time for that at this stage so it is still a work in progress.

At this point there has been quite a bit of setting the scene and reviewing content that was linked to the Nuffield summer project, now that comes to an end. Everything from here on in is related to the Nuffield work purely by using it as a starting point from which to attack other problems.

5 Pullbacks of Distributions, Definitions

5.1 Setup

In the previous sections we have developed several techniques related to the use of coordinate free distributions. These included how to deal with $d$’s, $i_v$’s and $\ast$’s, then the concept of the pushforward of a map was extended from vectors to these distributional $p$ forms. However, there is another map that we can extend to the setting of $p$ form distributions, that is the pullback map. In standard differential geometry for a smooth map between differentiable manifolds, $f : N \to M$, the pullback is the map $f^* : \Gamma \Lambda M \to \Gamma \Lambda N$.

This can also be extended to the distributional setting, using the definition below. However, a major difference between the distributional pullback and its pushforward counterpart is that there are two possible definitions for interaction of the pullback with the action of a distribution on a test form. Both these definitions are given below, they differ because of the properties of the map $f : N \to M$ from which the pullback is built.

**Definition 5.1.1.** For a map $f : N \to M$ the associated distributional pullback is given by $f^* : \Gamma_D \Lambda M \to \Gamma_D \Lambda N$

At this point a key definition is that of the pullback bundle.

**Definition 5.1.2.** Given the smooth maps $a : A \to M$ and $f : N \to M$ between manifolds. Then the pullback bundle is the manifold defined by,

$$A \times_{(a,f)} N = \{(z,x) \in A \times N | a(z) = f(x)\}.$$  \hspace{1cm} (5.1)

It can be denoted by either $a^* N$ of $f^* A$, though the second notation is more common.

As mentioned above, there are 2 possible definitions of the distributional pullback depending on how the manifolds are related. The first comes from considering the diagram below:

$$\begin{array}{ccc}
A^* A & \xrightarrow{a} & A \\
\downarrow f & & \downarrow a \\
N & \xrightarrow{f} & M
\end{array}$$
**Definition 5.1.3.** *Distributional Pullback 1:* For a distribution, $\Psi = a_\varsigma \alpha \in \Gamma_D \Lambda M$ and a map $f : N \to M$ the pullback, $f^\varsigma \Psi$ is given by,

$$f^\varsigma a_\varsigma \alpha = \hat{f}_\varsigma \hat{a}^* \alpha \quad (5.2)$$

The second definition is for the case when $f$ is a projection.

**Definition 5.1.4.** *Distributional Pullback 2:* For $\phi \in \Gamma_0 \Lambda N, \Psi \in \Gamma_D \Lambda M$, $f : N = U \times M \to M$, $f^\varsigma \Psi \in \Gamma_D \Lambda N$ and $f_\varsigma \phi \in \Gamma_0 \Lambda M$, because $N = U \times M$ $f_\varsigma$ represents a fiber integration. Then

$$\langle \phi, f^\varsigma \Psi \rangle_N = (-1)^r p \langle f_\varsigma \phi, \Psi \rangle_M, \quad (5.3)$$

here $r = \dim U$ and $p = |\alpha|$.

A fiber integration means that the pushforward integrates out some of the coordinates; for example if $f : U \times M \to M$ and $\phi \in \Gamma_0 \Lambda(U \times M)$ then

$$f_\varsigma \phi = \int_U \phi.$$

This can be seen in Figure 4. The two definitions given above can both be defined at the same time, in these circumstances they may always be equal. However, this is still uncertain; what is known is that in a specific set of circumstances they are equivalent.

**Theorem 4.** When both $f : N \to M$ is a projection and $\Psi = a_\varsigma \alpha$ then,

$$f^\varsigma a_\varsigma \alpha = f^\varsigma a_\varsigma \alpha. \quad (5.4)$$

### 5.2 Proof of Equivalence

Before the proof is given we have to introduce an integration convention to account for possible sign problems. The convention is that the fiber must always be integrated over first. In other words,

$$\int_N \phi = \int_{U \times M} \eta \wedge \chi = \int_M \eta \int_U \chi, \quad (5.5)$$

where $\phi \in \Gamma_0 \Lambda N = \Gamma_0 \Lambda(U \times M)$ and $\eta$ and $\chi$ are the parts of $\phi$ on $U$ and $M$ respectively.

**Proof.** The proof of Theorem 4 takes a bit of work and starts from taking $f$ s.t it is a projection and also using $\Psi = a_\varsigma \alpha$. This leads to the same diagram as before.

\[
\begin{array}{c}
U \times A \xrightarrow{\hat{a}} A \\
\downarrow f \\
N \xrightarrow{f} M
\end{array}
\]
Figure 4: A fibered manifold, $N = U \times M$, where $M$ is the base manifold and $U$ gives the fibers [4]. The link between the pushforward of a projection and fibre integration arises because a projection can be thought of as integrating out these fibres.

Here it has already been filled in that $f^*A = \{(z, u, x) \in A \times U \times M | a(z) = x\} = \{(u, z)\} = U \times A$, where we also require that $U \times M \subset N$ so we can say that $N = U \times M$. This allows us to make the assumption that $\forall \phi \in \Gamma_0 \Lambda N \exists \chi \in \Gamma_0 \Lambda^{top} U$ and $\eta \in \Gamma_0 \Lambda^{m-q} M$ s.t. $\phi = \eta \wedge \chi$. Also let $\int_U \chi = 1$. This isn’t completely true, in fact $\phi$ is expressible as the limit of expressions of the form given above. We should note also the degree of $\alpha$, $\alpha \in \Gamma \Lambda^p A$, ideg($\alpha, \alpha$) = $p$ and dim$U = r$.

16
Start from,

\[ \langle \phi, f^\natural a_\natural \alpha \rangle_{U \times M} = \langle \phi, \hat{f}^\natural \alpha \rangle_{U \times M} \]
\[ = \langle \hat{f}^\natural (\phi), \hat{a}^\natural \alpha \rangle_{U \times A} \]
\[ = \langle \hat{f}^\natural (\eta \wedge \chi), \hat{a}^\natural \alpha \rangle_{U \times A} \]
\[ = \int_{U \times A} \hat{f}^\natural (\eta) \wedge \hat{f}^\natural (\chi) \wedge \hat{a}^\natural (\alpha) \]
\[ = (-1)^p \int_{U \times A} \hat{f}^\natural (\eta) \wedge \alpha \wedge \chi \]
\[ = (-1)^p \left( \int_A \hat{a}^\natural (\eta) \wedge \alpha \right) \int_U \chi \]
\[ = (-1)^p \langle a^\natural (\eta), \alpha \rangle_A \]
\[ = (-1)^p \langle \eta, a_\natural \alpha \rangle_M \]
\[ = (-1)^p \langle f^\natural \phi, a_\natural \alpha \rangle_M. \]

Giving,

\[ \langle f^\natural \phi, a_\natural \alpha \rangle_M = (-1)^p \langle \phi, f^\natural a_\natural \alpha \rangle_N. \] (5.6)

Which is nothing but Definition 5.1.4, thus showing that in these special circumstances they are equivalent.

Also note that the distributional pullback still preserves degree so \(|f^\natural (\alpha)| = |\alpha|\).

### 5.3 Relation of Distributional Pullback to \(d\), \(i_v\) and \(\star\)

The properties proved below all come naturally from Definition 5.1.4. However we conjecture that because the two definitions are equivalent in a certain setting that these properties will hold more generally.

**Theorem 5.** For a distribution \(\Psi \in \Gamma_{\mathcal{D} \Lambda^* M}\), a test form \(\phi \in \Gamma_{0 \Lambda^* N}\), a map \(f : N \rightarrow M\) and the exterior derivatives on \(M\) and \(N\) are \(d_M\) and \(d_N\) respectively. For the second and third relationship, \(\star_M\) and \(\star_N\) are the star operator acting on \(M\) and \(N\) respectively and also \(v \in \Gamma T N\) and \(w \in \Gamma T M\) such that \(w|_f(p) = f_\natural(v)|_p\).

- The exterior derivative commutes with the distributional pullback,
  \[ f^\natural d_M \Psi = d_N f^\natural \Psi. \] (5.7)

- The internal contraction commutes with the distributional pullback,
  \[ f^\natural i_v \Psi = \phi_i \Psi. \] (5.8)

- The Hodge dual on different manifolds can be related by its interaction with the distributional pullback,
  \[ f^\natural \star_M \Psi = (-1)^{r(m+p)} \star_N f^\natural \Psi. \] (5.9)

where \(m = \dim M\), \(n = \dim N\) and \(p = |\Psi|\). Also as \(f\) is a projection from \(N = U \times M\) to \(M\), \(\dim U = r = n - m\) is also needed.
Proof. The relationships will be proved in turn; the same logic is used for each but it is instructive to include all the proofs as it gives an idea of how calculations are done using this language.

- For the \( d \) relationship start with
  \[
  \langle \phi, f^{s^2} d_M \Psi \rangle_N = (-1)^{(p+1)} \langle f \phi, d_M \Psi \rangle_M \\
  = (-1)^{(p+1)} (-1)^{p-m} \langle d_M f \phi, \Psi \rangle_M \\
  = (-1)^{(p+1)} (-1)^{p-m} (-1)^p \langle f d_N \phi, \Psi \rangle_M \\
  = (-1)^{r} (-1)^{p-m} \langle d_N \phi, f^{s^2} \Psi \rangle_N \\
  = (-1)^{n-m} (-1)^{p-m} (-1)^{p-n} \langle \phi, d_N f^{s^2} \Psi \rangle_N \\
  = \langle \phi, d_N f^{s^2} \Psi \rangle_N,
  \]
  giving the first result.

- The second result uses pretty much the same logic,
  \[
  \langle \phi, f^{s^2} i_w \Psi \rangle_N = (-1)^{(p-1)} \langle f \phi, i_w \Psi \rangle_M \\
  = (-1)^{(p-1)} (-1)^{p-m} \langle i_w f \phi, \Psi \rangle_M \\
  = (-1)^{(p-1)} (-1)^{p-m} \langle f i_w \phi, \Psi \rangle_M \\
  = (-1)^{(p-1)} (-1)^{p-m} (-1)^p \langle i_w \phi, f^{s^2} \Psi \rangle_M \\
  = (-1)^{-r} (-1)^{p-m} (-1)^{p-n} \langle \phi, i_w f^{s^2} \Psi \rangle_M \\
  = (-1)^{m-n} (-1)^{n+m} \langle \phi, i_w f^{s^2} \Psi \rangle_M \\
  = \langle \phi, i_w f^{s^2} \Psi \rangle_M,
  \]
  which gives the second result.

- The final result uses slightly different logic, specifically it uses the fact that the pushforward can be thought of as an integral. It is also important to note that \( *_M \) and \( *_N \) are the hodge dual on \( M \) and \( N \) respectively.
  \[
  \langle \phi, f^{s^2} *_M \Psi \rangle_N = (-1)^{(m-p)} \langle f \phi, *_M \Psi \rangle_M \\
  = (-1)^{(m-p)} (-1)^{*_M \phi} \langle \Psi, *_M f \phi \rangle_M \\
  = (-1)^{r(m-p)} (-1)^{*_M \phi} \langle \Psi, *_M f \phi \rangle_M
  \]
  Now comes the relevance of the pushforward being an integral, in other words \( f \phi \) integrates out the parts of \( \phi \) that don’t live on \( M \) so,
  \[
  *_M \int_{N \setminus M} \phi = \int_{N \setminus M} *_M \phi = \int_{N \setminus M} *_N \phi.
  \]
  This allows us to continue the train of thought from above.
  \[
  \langle \phi, f^{s^2} *_M \Psi \rangle_N = (-1)^{(m-p)} (-1)^{*_M \phi} \langle \Psi, *_M \phi \rangle_M \\
  = (-1)^{(m-p)} (-1)^{*_M \phi} \langle \Psi, (-1)^{*_N \phi} f^{s^2} \Psi \rangle_N \\
  = (-1)^{r(m)} (-1)^{*_M \phi} \langle \Psi, (-1)^{*_N \phi} f^{s^2} \Psi \rangle_N
  \]
  18
Now we need to care about the sign, \(|\star_M f_*\phi| = m - p\) and \(|\star_N \phi| = n - p\). So the sign at the front becomes,

\[
(-1)^{m+|\star_M f_*\phi|+|\star_N \phi|} = (-1)^{m+(m-p)+n-(n-p)p} = (-1)^{r+pn}.
\]

This gives the final result.

As was mentioned above these results have been proved using Definition 5.1.4, though it is conjectured that they are properties of the pullback that will hold more generally.

### 5.3.1 Uniqueness of Vectors

The second result in Theorem 5 raises some interesting questions, most noteworthy of which is “When is the vector \(f_*u\) defined?” The answer to this for the case of \(f\) being a projection is always. This is because all the vectors on \(M\) are also on \(N\) as \(M \subset N\). This can be expressed as the result below.

**Lemma 5.1.** For the second pullback definition, \(\forall v \in TN \exists w \in TM\) where \(w|_f = f_*v|_p\), such that

\[
f^s_i w a_\varsigma \alpha = i_v f^s_i a_\varsigma \alpha
\] (5.10)

is defined.

We can extend this to the case when \(\dim M > \dim N\), the relm of Definition 5.1.3. This follow from considering the diagram below.

![Diagram](image)

The splitting of a vector into pieces on \(TA\) and \(TN\) is again useful. Consider,

\[
f^s_i w a_\varsigma \alpha,
\] (5.11)

where \(w \in TM\) is decomposed as \(w = f_*v + a_*u\) with \(v \in TN\) and \(u \in TA\). Now we can re-express this to get an expression for \(f^s_i w a_\varsigma \alpha\).

\[
f^s_i w a_\varsigma \alpha = f^s_i (f_*v + a_*u) a_\varsigma \alpha
\]  
\[= f^s_i f_*v a_\varsigma \alpha + f^s_i a_*u a_\varsigma \alpha
\]  
\[= i_v f^s_i a_\varsigma \alpha + f^s_i a_\varsigma i_u \alpha
\]

So we have a way of dealing with \(i_w\) in Definition 5.1.3 that is unique up to the decomposition of \(w\).
6 The ∆ Conjecture

The Key result for this section is the proof, in two settings, of a nice conjecture about the ∆ distribution. This is the object that gives rise to the “δ-function”. ∆ ∈ Γ_D(Λ_Mx × M_y), where m = dimM and M_x and M_y are two identical copies of a manifold M. They are labeled distinctly for ease of identification and because one is parameterised with x coordinates and the other with y. They will prove very useful when developing the Green’s function techniques later on, in fact they are key to the entire construction. The form of ∆ involves the pushforward of a rather interesting map, ˆ∆ : M → M_x × M_y or in terms of coordinates ˆ∆ : x → (x, x). Using this map ∆ = ˆ∆ ς(1)， so ∆ is the distribution corresponding to the m dimensional surface that results when the point distribution 1 is pushed forward by ˆ∆. The ∆ conjecture, itself key to the Green’s function results explored later, is given below.

Conjecture 1. For a distribution Ψ ∈ Γ_D(Λ_My) the pullback of Ψ, π_y*Ψ, where π_y : M_x × M_y → M_y satisfies the following relation,

\[ \pi_x(\hat{\Delta} \cdot 1 \wedge \pi_y^*(\Psi)) = Id_{(y \to x)} \cdot \Psi \] (6.1)

Developing the setup necessary to go about proving the conjecture is the next step to take.

6.1 Setup

The setup used is an extension of the one that was discussed before. It is developed by considering a manifold M, generating the product manifold M_x × M_y using the ˆ∆ map introduced above. The next thing we need are two projection maps, π_x : M_x × M_y → M_x and π_y : M_x × M_y → M_y. There will also another manifold A which is linked to M_y through a map a : A → M_y, as well as this there will be two pullback manifolds, B = π_y*A and C = ∆*B. The diagram of relations is given below and will allow the proof of the ∆ conjecture to be formulated properly.

Where the form of the manifolds C and B are found using ∆ : x → (x, x), as being B = M_x × M_y × A/(π_y, a) = \{(x, y, z) ∈ M_x × M_y × A|y = a(z)\} = M_x × A and C = M × M_x × A/(\hat{\Delta}, (Id_x × a)) = \{(m, x, z) ∈ M × M_x × A|(m, m) = (x, a(z))\} = A.

The maps not marked on the diagram are:
a : A → M which takes w → a(w), this is for the A manifold on the left,
\[(a \times 1_a) \circ \hat{\Delta}_A : A \to M_x \times A \text{ or in terms of coordinates } w \mapsto (a(w), w),\]
\[(a \times a) \circ \hat{\Delta}_A : A \to M_x \times M_y \text{ which takes } w \mapsto (a(w), a(w)),\]
where \(\hat{\Delta}_A : A \to M_x \times M_y\) which in terms of coordinates takes \(w \mapsto (w, w)\).

This results in the diagram becoming:

\[
\begin{array}{ccc}
A & \longrightarrow & M_x \times A \\
\downarrow & & \downarrow \pi_a \\
M & \xrightarrow{\Delta} & M_x \times M_y \\
\downarrow & & \downarrow \pi_x \\
M_x & & M_y
\end{array}
\]

The \(\Delta\) Conjecture can be proved for the case of submanifold distributions. First it shall be proved for distributions of the form \(\Psi = a_\zeta \alpha\), then this result shall be used to show it is still true for the case \(\Psi = i_v a_\zeta \alpha\). After the second case, the result can be extended to the more general submanifold distribution case using the results we already have about the interaction of the distributional pullback, \(d\) and \(i_v\).

### 6.2 Proof of the \(\Delta\) conjecture for \(\Psi = a_\zeta \alpha\)

The first step is to look at the simplest case, \(\Psi = a_\zeta \alpha\).

**Proof.** To do this a lot of time will be spent chasing definitions for the maps around the commuting diagram. If we take \(\Psi = a_\zeta \alpha\) where \(\alpha \in \Gamma_D \Lambda^q A\). We want to show that \(\Psi\) satisfies the following equation,

\[
\pi_{x\zeta}(\Delta \wedge \pi_{y\zeta}^*(\Psi)) = \text{Id}_{(y\to x)\zeta} \Psi. \quad (6.2)
\]

This wedge product needs to be evaluated, which involves manipulating both terms to be on \(A\) before pushing forward to \(M_x \times M_y\) and then on to \(M_x\).

First look at \(\Delta = \hat{\Delta}_\zeta 1 \in \Gamma_D \Lambda^m (M_x \times M_y)\), on \(M\) \(\Delta\) goes to 1 and on \(A\) it is given by \(a^*(1) = 1\). So one term has been found already, and we now need to find \(\pi_{y\zeta}^*(a_\zeta(\alpha))\).

For a general \(\Psi\), \(\pi_{y\zeta}^*(\Psi)\) can be found by using

\[
(\text{Id}_x \times a)_\zeta(\pi_{a\zeta}^*(\alpha)) = \pi_{y\zeta}^*(a_\zeta(\alpha)), \quad (6.3)
\]
on \(M_x \times M_y\). This results in \(\pi_{a\zeta}^*(\alpha)\) on \(M_x \times A\) which is equivalent to \(\alpha\) on \(A\). Therefore, on \(A\), \(\Delta \wedge \pi_{y\zeta}^*(\Psi)\) becomes \(1 \wedge \alpha = \alpha\).

To get this on \(M_x \times M_y\), we need to push it forward with \((a \times a) \circ \hat{\Delta}_A\), giving \(((a \times a) \circ \hat{\Delta}_A)_\zeta(\alpha)\). Now pushed forward onto \(M_x\) we have \(a_\zeta(\alpha)\). Also, \(\text{Id}_{(y\to x)\zeta} \Psi = \text{Id}_{(y\to x)\zeta}(a_\zeta \alpha) = a_\zeta \alpha\).

So Conjecture 1 is satisfied for this case. \(\square\)
6.3 Proof of the $\Delta$ conjecture for $\Psi = i_v a_\varsigma \alpha$

If we take $\Psi = i_v a_\varsigma \alpha$ where $\alpha \in \Gamma_D \Lambda^q A$ and $v \in TM_y$. This is similar to the previous case. However, we now need to know how the pullback interacts with the internal contraction; this will require that $\exists \hat{v} \in T(\sigma(A) \times \sigma(A))$, where $\hat{v} = av_x + bv_y$ and $\pi_y(\hat{v}) = v = bv_y$. We know that

$$\pi_x \iota_\hat{v} \gamma = i_{\pi_x(\hat{v})} \pi_x \gamma = i_{av_x} \pi_x \gamma$$

and

$$\pi_y \iota_\hat{v} \gamma_D = i_{bv_y} \pi_y \gamma_D,$$

as this was proven earlier. There is also the relation that

$$(i_v + i_{v_x}) \Delta = 0,$$

where $v_x$ is the vector linked to $v_y$ by $Id(y \to x)s_y = v_x$. If $v$ has the required properties then the LHS of Eq (6.1) becomes;

$$Id(y \to x) i:\varsigma \alpha = i_{v_x} Id(y \to x) s(y)\varsigma \alpha.$$

Now for the RHS,

$$\Delta \wedge \pi_y (i_v a_\varsigma \alpha) = \Delta \wedge i_\phi \pi_y (a_\varsigma \alpha).$$

We will take the case when $\hat{v} = v_y$ i.e. $s_x(\hat{v}) = 0$.

$$0 = \pi_x i_{v_x}(\Delta \wedge \pi_y a_\varsigma \alpha)$$

$$= \pi_x (i_v \Delta \wedge \pi_y a_\varsigma \alpha + \Delta \wedge i_v \pi_y a_\varsigma \alpha)$$

$$= \pi_x (-i_v \Delta \wedge \pi_y a_\varsigma \alpha + \Delta \wedge i_v \pi_y a_\varsigma \alpha)$$

Dealing only with the first term we have,

$$\pi_x (-i_v \Delta \wedge \pi_y a_\varsigma \alpha) = -\pi_x i_v (\Delta \wedge \pi_y a_\varsigma \alpha) = -i_v \pi_x (\Delta \wedge \pi_y a_\varsigma \alpha).$$

This is because $i_v \pi_y a_\varsigma \alpha = 0$. Putting all this together gives that

$$\pi_x (\Delta \wedge i_v \pi_y a_\varsigma \alpha) = i_v \pi_x (\Delta \wedge \pi_y a_\varsigma \alpha).$$

This reduces the problem the case considered before so we know that this gives us,

$$\pi_x (\Delta \wedge i_v \pi_y a_\varsigma \alpha) = i_v Id(y \to x) \varsigma \alpha.$$  

Which gives us the required result. This will also hold for a $\Psi$ that included multiple $i_v$'s, of the correct form, and $d$'s because these will all hop across in the same way, only giving a sign change based on the degree of $\Delta$ but $|\Delta| = m = dim M$ and $ideg \Delta = 0$. This ensures that there won't be any sign change. Now that the $\Delta$ conjecture has been dealt with, we are in a position to start considering the physics applications of this formalism.
Green’s functions

Green’s functions are incredibly useful tools for physicists, who use them to find solutions to certain linear inhomogeneous partial differential equations. These techniques are very well developed in the standard setting. However, in the setting that we have been considering, coordinate free differential geometry, these haven’t been developed so we need to spend some time setting up the problem. This will involve utilising a lot of the tools we have developed earlier. Consider a differentiable manifold \( M \), two copies of it labeled \( M_x \) and \( M_y \), and the cartesian product of these two copies \( M_x \times M_y \).

\[
\begin{array}{ccc}
M_x & \xleftarrow{\pi_x} & M_x \times M_y \xrightarrow{\pi_y} M_y \\
\Delta & & \\
\end{array}
\]

Now consider a linear operator \( L \) on \( M \) that acts on \( \Psi \in \Gamma_D \Lambda \) giving \( \Phi \in \Gamma_D \Lambda M \).

\[
L[\Psi] = \Phi. \quad (7.1)
\]

There will be a corresponding operator \( L_x \) on \( M_x \times M_y \) which only acts on the \( M_x \) part of an object. As \( M_x \) is a copy of \( M \) on it \( L_x \) would just be called \( L \). This means that we can link \( L \) to \( L_x \) using,

\[
L \circ \pi_x = \pi_x \circ L_x. \quad (7.2)
\]

In the case of the pushforward representing a fiber integration this is a result similar to that for the hodge dual in Section 5.2, and can be generalised in a similar manner.

The next step is to define a Green’s distribution \( G \in \Gamma_D \Lambda(M_x \times M_y) \). This Green’s distribution will satisfy the equation,

\[
L_x[G] = \hat{\Delta}^1. \quad (7.3)
\]

This then gives the solution \( \Psi \) of Equation (7.1) as

\[
\Psi = \pi_x(\mathcal{G} \wedge \pi^*_y \Phi). \quad (7.4)
\]

That this is a valid solution can easily be shown.

\[\textbf{Proof.}\]

\[
L[\Psi] = L[\pi_x(\mathcal{G} \wedge \pi^*_y \Phi)] \\
= \pi_xL_x[(\mathcal{G} \wedge \pi^*_y \Phi)] \\
= \pi_xL_x[\mathcal{G}] \wedge \pi^*_y \Phi \\
= \pi_x(\hat{\Delta}^1 \wedge \pi^*_y \Phi) \\
= \Phi.
\]

Here we have used the above properties of \( L \) as well as the \( \Delta \) conjecture, Conjecture 1, for the final step. The use of the \( \Delta \) conjecture may seem to limit the applications of these methods. However, in the remaining sections only submanifold distributions will be used so that Conjecture 1 holds. \( \square \)
The use of Conjecture 1 in the above proof nicely shows the main motivation for that conjecture, which was developed for the purposes of the green function work now being explored.

7.1 Green’s Functions in Electrodynamics

A prime example of an area of physics where distributions are used profusely is electrodynamics. Here the Green’s functions techniques are typically used to find the electromagnetic potentials that give rise to the fields, [6]. These potentials are known as the Liénard-Wiechert potentials, which can be calculated using the methods developed here. However, that was not part of the project so the results will just be quoted when needed.

![Figure 5: The worldline of a charge intersecting a backwards light cone.](image)

The key equations of electrodynamics are Maxwell’s equations which have the form,

\[ d \ast A = 0, \]  
\[ (7.5) \]

and

\[ d \ast dA = \ast J, \]  
\[ (7.6) \]
where the first equation signifies that we are in the Lorentz gauge. We would expect the potential $A$ to have the form,

$$A = \pi_x (G \wedge \pi_y J), \quad (7.7)$$

In terms of the Green’s function $G$ which would be expected to solve the equation,

$$d \ast dG = \ast \Delta. \quad (7.8)$$

However, it turns out that $A$ found in this way is not guaranteed to satisfy Equation (7.5), or in other words $A$ doesn’t need to be in the Lorentz gauge. This means that the natural equation for $A$ to solve is no longer Maxwell’s equations but their generalisation, known as the Laplace-de Rham equation or the Laplace-Beltrami equation, Equation (7.9).

$$\Box A = \mathcal{J}, \quad (7.9)$$

where $\Box = d\delta + \delta d$, with $\delta$ being the co-derivative which is related to the standard $d$ by $\delta = (-1)^{nk+n+1}s \ast d\ast$. Here, $n$ is the dimensions of the space, $k$ is the degree of the form the operator is acting on and $s$ is related to the signature of the metric, for Lorentzian manifolds $s = -1$. For our purposes, $A \in \Gamma_D \Lambda^1 M$ giving $n = 4$ and $k = 1$ so $\delta = \ast d\ast$. This leads to Equation (7.9) becoming,

$$(d \ast d \ast + \ast d \ast d)A = \mathcal{J}. \quad (7.10)$$

This will have a Green’s function $G$ associated with it which solves the equation,

$$(d \ast d \ast + \ast d \ast d)G = \Delta = \hat{\Delta} \chi. \quad (7.11)$$

The commuting diagram for this is shown below.

\[\begin{array}{ccc} N \times \mathbb{R} & \xrightarrow{\iota_0} & M_x \times \mathbb{R} \xrightarrow{\pi_y} \mathbb{R} \\ h & \downarrow & f \\ N \times M_y & \xrightarrow{\iota} & M_x \times M_y \xrightarrow{\pi_y} M_y \\ \pi_x & \downarrow & \hat{\Delta} \\ M_x & \xrightarrow{\pi} & M \\
\end{array}\]

Where $N = \{ z \in M | g(z, z) = 0, z^0 > 0 \}$ and $\iota: (z, y) \rightarrow (z + y, y)$, therefore $\iota$ doesn’t affect things that are in terms of $y$. It is useful to note that the manifold $N$ is a cone and it represents the backwards lightcone on which $G$ has its support, shown in Figure 5. It is a 7 dimensional manifold with the constraint that $z^0 = |z| = z^1dz^1 + z^2dz^2 + z^3dz^3$. The case to be considered first is that of a charged monopole where we know that,

$$G = \iota_\chi \left( \frac{dz^{123}}{|z|} \right). \quad (7.12)$$

This is the standard result translated into coordinate free notation.
Theorem 6. Properties of $G$: The Green’s function $G$ has the following properties

1. $\ast_x G = \ast_y G$ or in other words $G$ is symmetric in $x$ and $y$.
2. $dG = 0$ or using $d = d_x + d_y$ we have $d_x G = -d_y G$.
3. Analogous to 2 above $i_{(v_x + v_y)} G = 0$ or in other words $i_{v_x} G = -i_{v_y} G$.

These are all related to the $x, y$ symmetry of $G$ and in fact the first property in old money would be shown by writing $G(x - y)$. In the following proof the shorthand notation $dx^a$ will be used to stand for $dx^a \wedge dx^b$.

Proof. The 3 properties all revolve around the symmetries of $G$,

I $\ast_x G = \ast_y G$:

To see this act $G$ on a test function $\phi \in \Gamma_0 \Lambda(M_x \times M_y)$ and then consider the degrees of terms which make up $\phi$. We know that $\ast_x \phi$ will have degree 4 but we don’t know about $\deg_x \phi$ and $\deg_y \phi$, other than that $\deg_x \phi = \deg_y \phi$, so we will have to take each degree in turn.

(a) $\deg_x \phi = 1 = \deg_y \phi$: So take $\phi = dx^1 \wedge dy^1$ then;

\[
\ast_x \phi = dx^023 \wedge dy^1
\]

\[
i^\ast(\ast_x \phi) = (dz^0 + dy^0) \wedge (dz^2 + dy^2) \wedge (dz^3 + dy^3) \wedge dy^1
\]

\[
i^\ast(\ast_x \phi) \wedge \frac{dz^{123}}{|z|} = (dy^0 \wedge dy^1 \wedge dy^2 \wedge dy^3 \wedge dz^1 \wedge dz^2 \wedge dz^3)(\frac{1}{|z|}).
\]

Now look at $\varphi = dx^2 \wedge dy^2$

\[
\ast_y \varphi = dx^2 \wedge dy^{013}
\]

\[
i^\ast(\ast_y \varphi) = (dz^2 + dy^2) \wedge dy^0 \wedge dy^1 \wedge dy^3
\]

\[
i^\ast(\ast_y \varphi) \wedge \frac{dz^{123}}{|z|} = \frac{1}{|z|}(dy^0 \wedge dy^1 \wedge dy^2 \wedge dy^3 \wedge dz^1 \wedge dz^2 \wedge dz^3)
\]

so we have that

\[
i^\ast(\ast_y \varphi) \wedge \frac{dz^{123}}{|z|} = i^\ast(\ast_x \phi) \wedge \frac{dz^{123}}{|z|}.
\]

Now the test function is a general 2 form so it will contain both the terms we looked at above so at least for those two terms the test function is symmetric. Now an analogous calculation can be done for any two form of the type used above, giving us that the test function is symmetric. So $\forall \phi$ such that $|\phi| = 2$

\[
(\phi, \ast_y G)_{M_x \times M_y} = (\phi, \ast_x G)_{M_x \times M_y}.
\]

(b) The next step is to do the same for $\deg_x \phi = 2 = \deg_y \phi$. Take $\phi = dx^{01} \wedge dy^{01}$,

\[
\ast_x \phi = dx^{23} \wedge dy^{01}
\]

\[
i^\ast(\ast_x \phi) = (dz^2 + dy^2) \wedge (dz^3 + dy^3) \wedge dy^0 \wedge dy^1
\]

\[
i^\ast(\ast_x \phi) \wedge \frac{dz^{123}}{|z|} = (dy^0 \wedge dy^1 \wedge dy^2 \wedge dy^3 \wedge dz^1 \wedge dz^2 \wedge dz^3)(\frac{1}{|z|})
\]

But $i^\ast(\ast_y (dx^{23} \wedge dy^{23})) = (dz^2 + dy^2) \wedge (dz^3 + dy^3) \wedge dy^0 \wedge dy^1$

\[
\Rightarrow i^\ast(\ast_x (dx^{01} \wedge dy^{01})) \wedge \frac{dz^{123}}{|z|} = i^\ast(\ast_y (dx^{23} \wedge dy^{23})) \wedge \frac{dz^{123}}{|z|}.
\]
The same logic as above extends this to a general $\phi$ such that $|\phi| = 4$ form again giving,

$$\langle \phi, \ast_y \mathcal{G} \rangle_{M_x \times M_y} = \langle \phi, \ast_x \mathcal{G} \rangle_{M_x \times M_y}.$$  

(c) $\deg_x \phi = 3 = \deg_y \phi$

$$\phi = dx^{013} \wedge dy^{013},$$

$$\ast_x (\phi) = dx^2 \wedge dy^{013},$$

$$i^*(\ast_x (\phi)) = (dz^2 + dy^2) \wedge dy^{013},$$

$$i^*(\ast_x (\phi)) \wedge \frac{dz^{123}}{|z|} = dy^0 \wedge dy^1 \wedge dy^2 \wedge dy^3 \wedge \frac{dz^{123}}{|z|}.$$  

However $i^*(\ast_y (dx^{012} \wedge dy^{012})) = (dz^0 + dy^0) \wedge (dz^1 + dy^1) \wedge (dz^2 + dy^2) \wedge dy^3$

$$\therefore i^*(\ast_y (dx^{012} \wedge dy^{012})) \wedge \frac{dz^{123}}{|z|} = i^*(\ast_x (dx^{013} \wedge dy^{013})) \wedge \frac{dz^{123}}{|z|}.$$  

Both these terms would be present in our general 6 form so we have that $\ast_y \phi = \ast_x \phi$, which leads to

$$\langle \phi, \ast_y \mathcal{G} \rangle_{M_x \times M_y} = \langle \phi, \ast_x \mathcal{G} \rangle_{M_x \times M_y},$$

for $|\phi| = 6$.

(d) $\deg_x \phi = 4 = \deg_y \phi$. All degree 8 forms are the same up to multiplication by a scalar so there is only one term that we need to look at.

$$\phi = dx^{0123} \wedge dy^{0123},$$

$$\ast_x \phi = dy^{0123},$$

$$i^*(\ast_x \phi) = dy^{0123},$$

$$i^*(\ast_x \phi) \wedge \frac{dz^{123}}{|z|} = dy^{0123} \wedge \frac{dz^{123}}{|z|},$$

$$\ast_y \phi = dx^{0123},$$

$$i^*(\ast_y \phi) = (dz^0 + dy^0) \wedge (dz^1 + dy^1) \wedge (dz^2 + dy^2) \wedge (dz^3 + dy^3),$$

$$i^*(\ast_y \phi) \wedge \frac{dz^{123}}{|z|} = dy^{0123} \wedge \frac{dz^{123}}{|z|},$$

$$\Rightarrow i^*(\ast_y \phi) \wedge \frac{dz^{123}}{|z|} = i^*(\ast_x \phi) \wedge \frac{dz^{123}}{|z|}.$$  

This gives that for a degree 8 test form,

$$\langle \phi, \ast_y \mathcal{G} \rangle_{M_x \times M_y} = \langle \phi, \ast_x \mathcal{G} \rangle_{M_x \times M_y}.$$
(e) Finally deg, φ = 0 = deg, φ.

\[ \phi = \phi_0 \]

\[ \star_x \phi = \phi_0 dx \wedge dy \]

\[ i^*(\star_x \phi) = i^*(\phi_0)(dz^0 \wedge dy^0) \wedge (dz^1 + dy^1) \wedge (dz^2 + dy^2) \wedge (dz^3 + dy^3) \]

\[ i^*(\star_x \phi) \wedge \frac{dz^T}{|z|} = i^*(\phi_0)dy \wedge dy^1 \wedge dy^2 \wedge dy^3 \wedge \frac{dz^T}{|z|} \]

\[ \star_y \phi = \phi_0 dy \wedge dx \]

\[ i^*(\star_y \phi) \wedge \frac{dz^T}{|z|} = i^*(\phi_0)dy \wedge dx \wedge dy^0 \wedge dy^1 \wedge dy^2 \wedge dy^3 \wedge \frac{dz^T}{|z|} \]

\[ i^*(\star_y \phi) \wedge \frac{dz^T}{|z|} = i^*(\star_x \phi) \wedge \frac{dz^T}{|z|} \]

This gives the result for |φ| = 0 forms,

\[ \langle \phi, \star_y G \rangle_{M_x \times M_y} = \langle \phi, \star_x G \rangle_{M_x \times M_y} \]

Now that we have shown the result to be true for all possible degrees of φ we have the desired result that G is symmetric under the action of hodge star’s acting on the x or y coordinates. In other words,

\[ \star_y G = \star_x G. \]

II \( d_2 G = -d_y G \): Start by looking at the action of \( dG \) on a test form, \( \phi \).

\[ \langle \phi, dG \rangle_{M_x \times M_y} = \langle d\phi, G \rangle_{M_x \times M_y} \]

\[ = \langle i^*(d\phi), \frac{dz^T}{|z|} \rangle_{N \times M_y} \]

\[ = \int i^*(d\phi) \wedge \frac{dz^T}{|z|} \]

\[ = \int di^*(\phi) \wedge \frac{dz^T}{|z|} \]

\[ = \int d(i^*(\phi) \wedge \frac{dz^T}{|z|}) \]

\[ = 0 \]

\[ \Rightarrow \int i^*(d_x \phi) \wedge \frac{dz^T}{|z|} = -\int i^*(d_y \phi) \wedge \frac{dz^T}{|z|}. \]

Or in other words,

\[ \langle \phi, d_y G \rangle_{M_x \times M_y} = -\langle \phi, d_x G \rangle_{M_x \times M_y} \]

\[ \therefore d_y G = -d_x G. \]
For this the properties of $i$ need to be used. $i: N \times M_y \rightarrow M_x \times M_y$ or in other words $(z, y) \mapsto (z + y, y)$ so on a vector $i_* v_y = v_x + v_y$. Now the desired result can be shown by considering the action of $\mathcal{G}$ on a test form $\phi$.

\[ \langle \phi, i_* \mathcal{G} \rangle_{M_x \times M_y} = \langle \phi, i_* (i_* \phi, \mathcal{G}) \rangle_{M_x \times M_y} \]
\[ = \langle \phi, (i_* (i_* \phi), \mathcal{G}) \rangle_{M_x \times M_y} \]
\[ = \langle \phi, (i_* (i_* \phi), \mathcal{G}) \rangle_{M_x \times M_y} \]
\[ = \langle \phi, (i_* (i_* \phi), \mathcal{G}) \rangle_{M_x \times M_y} \]
\[ = 0. \]

Which gives the desired result.

As $\mathcal{G}$ is the Green’s function for the Laplace-Beltrami equation, (7.11), $\mathcal{G}$ can be used to find $A$ using,

\[ A = \pi_x(\mathcal{G} \wedge \pi_y^c \mathcal{J}). \quad (7.13) \]

This is just a use of the formalism that was developed above but for the specific setting of the Laplace-Beltrami equation. We also know that we can do a star pivot on $\ast_y \mathcal{G} \wedge \pi_y^c \mathcal{J}$.

Now we turn to the key theorem, that the potential, $A$ which solves the Laplace-Beltrami equation, Equation (7.9), will also solve Maxwell’s equations, Equation (7.6), under certain conditions. In fact it turns out that the potential will also satisfy the Lorentz gauge condition, Equation (7.5), under the same condition. This means that we effectively get the Lorentz gauge condition for free. The first thing to formulate is a rather nice lemma.

**Lemma 7.1.** For $A = \pi_x(\mathcal{G} \wedge \pi_y^c \mathcal{J})$ then $d \ast A$ is given by

\[ d \ast A = \pi_x(\mathcal{G} \wedge \pi_y^c (d \ast \mathcal{J})). \quad (7.14) \]

This lemma is a very useful result as it is applicable to any solution to the Laplace-Beltrami equation.

**Proof.** Start from $A = \pi_x(\mathcal{G} \wedge \pi_y^c \mathcal{J})$ Then look at $d \ast A$ and use the properties of the pushforward, pullback and $\mathcal{G}$.

\[ d \ast A = d \ast \pi_x(\mathcal{G} \wedge \pi_y^c \mathcal{J}) \]
\[ = d\pi_x \ast_x (\mathcal{G} \wedge \pi_y^c \mathcal{J}) \]
\[ = d\pi_x \ast_x (\ast_y \mathcal{G} \wedge \pi_y^c \mathcal{J}) \]
\[ = d\pi_x \ast_x (\ast_y \mathcal{G} \wedge \pi_y^c \mathcal{J}) \]
\[ = \pi_x d_x (\mathcal{G} \wedge \ast_y \pi_y^c \mathcal{J}) \]
\[ = \pi_x d_x (\mathcal{G} \wedge \pi_y^c (\ast \mathcal{J})) \]
\[ = \pi_x (d_x \mathcal{G} \wedge \pi_y^c (\ast \mathcal{J})). \]
Here we need to think about
\[
0 = \pi_x \mathcal{G} \land \pi_y^*(*J)
\]
\[
= \pi_x (\delta_y \mathcal{G} \land \pi_y^*(*J) + \mathcal{G} \land \delta_y \pi_y^*(*J))
\]
\[
= \pi_x (-\delta_x \mathcal{G} \land \pi_y^*(*J) + \mathcal{G} \land \delta_y \pi_y^*(*J))
\]
\[
\Rightarrow \pi_x (\delta_x \mathcal{G} \land \pi_y^*(*J) = \pi_x (\mathcal{G} \land \delta_y \pi_y^*(*J))
\]
which in turn leads to a realisation about \(d \star A\).
\[
d \star A = \pi_x (\mathcal{G} \land \delta_y \pi_y^*(\star J))
\]
\[
= \pi_x (\mathcal{G} \land \pi_y^*(d \star J)).
\]
\[\square\]

**Theorem 7.** Let \(\Box A = J\) then if \(d \star J = 0\) this implies that
1. \(d \star A = 0\).
2. and \(d \star dA = \star J\).

**Proof.** We can use Lemma 7.1 as we know the form of \(d \star A\), so if \(d \star J = 0\) we get trivially to the desired result,
\[
d \star A = \pi_x (\mathcal{G} \land \pi_y^*0) = 0
\]
for \(7.15\).

Now that we have 1 we can show 2 fairly easily. We know that \((d \star d \star + \star d \star d)A = J\) and we have just proved that \(d \star A = 0\) Putting these together gives,
\[
J = (d \star d \star + \star d \star d)A = \star d \star dA,
\]
for \(7.16\) which gives the desired result. \[\square\]

For the monopole \(J = \star c_\zeta (1^D)\), giving \(A = h_\zeta (c_0 \left(\frac{dz_{123}}{|z|}\right))\). Here \(|J| = 1\) and \(|A| = 1\) which agrees with the constraint that \(\Box\) preserves degree.

## 8 The Dipole

The next logical step after the monopole is to look at the dipole. Originally this was to be the main aim of the project. However, it proved rather complicated, with more groundwork needing to be done. Another downside is that there would have been a fair bit of overlap with the products of distributions work done in the Nuffield summer project, so it was desirable to avoid the overlap as much as possible. The dipole still posed a very interesting problem, therefore it was the natural extension of the project topic. The following conjecture, Conjecture 2, is based on a possible dipole source that looks interesting. There is still a lot of work that needs to be done, which involves answering such questions as “What do we mean by a dipole?”. The work presented below is all the progress that has been made so far, though hopefully more will follow when the opportunity presents itself.
Conjecture 2. A dipole potential which satisfies Maxwell’s equations is given by

\[ A_{dp} = \star d\hat{w} \star A_{LW}, \]  

where \( A_{LW} \) is the standard Liénard-Wiechert potential, and \( \hat{w} \) is a vector defined everywhere. It is related to the vector \( w \) on the world line of the dipole through parallel transport along the light cone.

This potential obviously satisfies the Lorentz gauge condition, Equation (7.5),

\[ d \star \star d\hat{w} \star A_{LW} = 0. \]  

The next thing is to find \( J \) using \( A_{LW} = \pi_x (G \land \pi_y^c J_{mp}) \).

\[ \star d\hat{w} \star A_{LW} = \star d\hat{w} \star \pi_x (G \land \pi_y^c J_{mp}) \]

\[ = \star d\hat{w} \pi_x \star \pi_x (G \land \pi_y^c J_{mp}) \]

\[ = \star d\hat{w} \pi_x \star \pi_x (G \land \pi_y^c J_{mp}) \]

\[ = \star d\hat{w} \pi_x \star \pi_x (G \land \pi_y^c J_{mp}) \]

\[ = \star d\pi_x \pi_y (G \land \pi_y^c J_{mp}) \]

\[ = \star d\pi_x \pi_y (G \land \pi_y^c J_{mp}) \]

\[ = \star d\pi_x \pi_y (G \land \pi_y^c J_{mp}) \]

\[ = \star d\pi_x \pi_y (G \land \pi_y^c J_{mp}) \]

Now we need to swap the \( i\hat{w}_y \) over to the other side of the wedge product.

\[ 0 = \pi_x \pi_y (G \land \pi_y^c J_{mp}) \]

\[ = \pi_x (i\hat{w}_y G \land \pi_y^c J_{mp} + G \land i\hat{w}_y \pi_y^c J_{mp}) \]

\[ = \pi_x (-i\hat{w}_y G \land \pi_y^c J_{mp} + G \land \pi_y^c (i\hat{w}_y J_{mp})) \]

\[ = \pi_x ((i\hat{w}_y G \land \pi_y^c J_{mp}) = \pi_x (G \land \pi_y^c (i\hat{w}_y J_{mp})) \]

This is then used to continue the above expression.

\[ \star d\hat{w} \star A_{LW} = \star d\pi_x \pi_y (i\hat{w} \star J_{mp}) \]

\[ = \pi_x \star x (d_x \pi_y (i\hat{w} \star J_{mp})) \]

\[ = \pi_x \star x (d_x \pi_y^c (i\hat{w} \star J_{mp})) \]

\[ = \pi_x \star x (d_x \pi_y^c (i\hat{w} \star J_{mp})) \]

\[ = \pi_x \star x (d_x \pi_y (i\hat{w} \star J_{mp})) \]

\[ = \pi_x \star x (d_x \pi_y^c (i\hat{w} \star J_{mp})) \]

\[ = \pi_x \star x (d_x \pi_y (i\hat{w} \star J_{mp})) \]

31
Which gives us that the dipole current will be of the form,

\[ J_{dp} = \star d i_w \star J_{mp}. \] (8.3)

Where \( w \) is used instead of \( \hat{w} \) because \( J_{dp} \) is only defined along the world line so we can use the vector that is defined on the world line. This current is of the correct degree, degree 1, which is the same as \( J_{mp} = \star C_\varsigma(1^D) \).

There are several gaps in this calculation that still need to be dealt with. These include how to deal with the \( d \) in \( d(\mathcal{G} \wedge i_w \pi_y(J)) \). There are several ideas about how to deal with this, which were discussed earlier in the products of distributions section. Another issue, in fact the main stumbling block, is why \( w \) is changed to \( \hat{w} \). In other words why do we have to pick the parallel transport of \( w \) along the light cone. The answer to this last point is unknown and work is still being conducted to try and find out exactly what we need to do with the vector and why.

9 Conclusion

The work carried out in the preceding sections was all done with the aim of translating established techniques into a new formalism, and in this respect it parallels the work of Heaviside in rewriting Maxwell’s equations and their solutions in the established vector calculus form. The focus was more on the solutions, the rewriting of the equations being the standard one when interpreting electrodynamics problems using differential geometry. Though to get this rewriting of the solutions, several techniques from differential geometry had to be extended to apply to a geometric interpretation of distributions; such a thing is commonly referred to as distributional geometry [7]. These techniques ranged from the pushforward map and how it can be used on a form if viewed as a fiber integration, through the pullback and how it interacts with \( d, i_v \) and \( \star \) to under what circumstances it is meaningful to wedge together two distributions. There were several results that needed to be proven in each case; once these were taken care of they could be applied to specific problems, such as Conjecture 1. This all lead up to being able to approach that well established work horse of theoretical physics, the method of Green’s functions, from a new angle. This reinterpretation of Green’s functions has potentially wide ranging applications as there are many areas of physics where the problems are more naturally expressed in a coordinate free manner.

The test of this reinterpretation was in calculating the electromagnetic potential for the monopole, this was very successful and along the way several more general results were found to aid in finding solutions to Maxwell’s equations. The next step was to consider an open problem, the dipole, to see what this reinterpretation could say about its solution. Though the problem could not be solved in its entirety a solution with certain gaps was found which matched the solution expected. This solution was based off a current that was conjectured to have the required properties to be a solution to Maxwell. The gaps do not seem too limiting, in fact there is already a possible way to fill one of them, this comes from some of the work on wedge products of distributions, which seems promising. As for the remaining problems, though there is no starting point for solving them currently it is hoped that with further work the correct approach
will become apparent. Once the Dipole calculation has been shored up it will be possible to approach higher order terms in the multipole expansion. Other future work could involve extending some of the techniques developed to the realm of Colombeau generalized functions, one of the extensions to the linear distribution theory [8]. This would enable applications to the solution of non-linear problems, including those from general relativity. Though the Green’s functions techniques will probably not extend, there have not be any claims made that they should, enough of the other techniques should that it would still be of use. This all goes to show that electrodynamics can still be viewed as an active and exciting field of research over a hundred years after its inception.

References


A Basic Properties of Distributions

The first of the relations given in Section 2 is about the relation between the exterior derivative and distributions.

**Result A.1.**

\[ \langle \varphi, d(\alpha^D) \rangle_M = \varepsilon(p,n) \langle \varphi, \alpha^D \rangle_M \]

Aim: Find \( \varepsilon(p,n) \) such that \( (d\alpha)^D = d(\alpha^D) \)

Where \( \alpha \in \Gamma \Lambda^p M \), \( \varphi \in \Gamma_0 \Lambda^{n-p-1} M \) and \( \dim M = n \).

\[ \langle \varphi, (d\alpha)^D \rangle_M = \int_M \varphi \wedge d\alpha \]

and

\[ \langle \varphi, (d\alpha)^D \rangle_M = \varepsilon(p,n) \langle \varphi, \alpha^D \rangle_M = \varepsilon(p,n) \int_M d\varphi \wedge \alpha \]
Remembering that
\[d(\varphi \wedge \alpha) = d\varphi \wedge \alpha + (-1)^{n-p-1}\varphi \wedge d\alpha\]
\[\Rightarrow \varphi \wedge d\alpha = \frac{1}{(-1)^{n-p-1}}[d(\varphi \wedge \alpha) - d\varphi \wedge \alpha]\]

This leads to
\[\langle \varphi, (d\alpha)^D \rangle_M = \int_M \varphi \wedge d\alpha = (-1)^{p+1-n} \int_M [d(\varphi \wedge \alpha) - d\varphi \wedge \alpha] \]
\[= (-1)^{p+1-n} \int_M d(\varphi \wedge \alpha) + (1)^{p-n} \int_M d\varphi \wedge \alpha \] 
\[= (-1)^{p+1-n} \int_M \varphi \wedge \alpha + \frac{(1)^{p-n}}{\varepsilon(p, n)} \langle \varphi, d(\alpha^D) \rangle_M \]
\[= \langle \varphi, d(\alpha^D) \rangle_M \iff \varepsilon(p, n) = (1)^{p-n} \]
\[\therefore \varepsilon(p, n) = (-1)^{p-n}.\]

The next relation is the analogous result for internal contractions.

Result A.2.
\[\langle \varphi, i_v(\alpha^D) \rangle_M = \varepsilon(p, n) \langle i_v\varphi, \alpha^D \rangle_M \]
where \(\varepsilon(p, n) = (-1)^{p-n}.\)

The proof is very similar so wasn’t included. It is useful to note that because of the degree of \(\varphi\) the sign can be reinterpreted,
\[(-1)^{p-n} = (-1)^{1-|\varphi|}.\]

The final result is about the Hodge dual.

Result A.3.
\[\langle \varphi, *(\alpha^D) \rangle_M = (-1)^\chi \langle *\varphi, (\alpha) \rangle_M \]

we also want that \(*(\alpha)^D = (*\alpha)^D\)
\[\langle \varphi, (*\alpha)^D \rangle_M = \int_M \varphi \wedge *\alpha \]
\[= \int_M \alpha \wedge \varphi \]
\[= (-1)^{||\alpha||*|\varphi|} \int_M *\varphi \wedge \alpha \]
\[= (-1)^{||\alpha||*|\varphi|} \langle *\varphi, (\alpha) \rangle_M \]

Giving us that
\[\langle \varphi, *(\alpha)^D \rangle_M = (-1)^{||\alpha||*|\varphi|} \langle *\varphi, (\alpha) \rangle_M \]

or in other words \(\chi = ||\alpha|| \ast |\varphi|\).